Northwest Academy of Sciences. Mathematical Circle.

Year 5. Problem Set 120 (2009-2010 school year).

1. Suppose $P(x)$ is a polynomial. Prove that the remainder of the division of $P(x)$ by $x - c$ is $P(c)$.

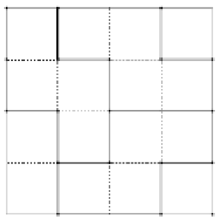
2. What is the remainder of $x^{243} + x^{81} + x^{27} + x^9 + x^3 + x$ when divided by $x - 1$.

3. $P(x) = x^{2n} + a^{2n}$. Prove that if $a \neq 0$ then $P(x)$ is not divisible by $x + a$ and is not divisible by $x - a$.

4. Prove that $a\frac{(x-b)(x-c)}{(a-b)(a-c)} + b\frac{(x-c)(x-a)}{(b-c)(b-a)} + c\frac{(x-a)(x-b)}{(c-a)(c-b)} = x$

5. A triangle (with non-zero area) is constructed with the lengths of the sides chosen from the set: $\{2; 3; 5; 8; 13; 21; 34; 55; 89; 144\}$ Show that this triangle must be isosceles. (A triangle is isosceles if it has at least two sides the same length.

6. .
Determine the greatest number of figures congruent to  that can be placed in a 9×9 grid (without overlapping), such that each figure covers exactly 4 unit squares. The figures can be rotated and flipped over. For example, the picture below shows that at least 3 such figures can be placed in a 4×4 grid.



7. N teams participated in a national basketball championship in which every two teams played exactly one game. Of the N teams, 251 are from California. It turned out that a California team, Alcatraz, is the unique California champion (Alcatraz won more games against California teams than any other team from California). However, Alcatraz ended up being the unique loser of the tournament because it lost more games than any other team in the nation!
What is the smallest possible value for N?